Nonlinear Optics as a Path to High-Intensity Circular Machines

S. Nagaitsev, A. Valishev (Fermilab),
and V. Danilov (ORNL)

Dec 17, 2010

COLLIDING BEAMS: PRESENT STATUS; AND THE SLAC PROJECT*

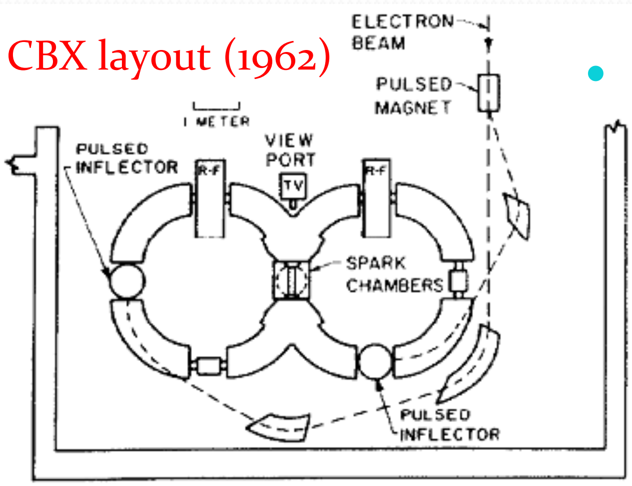
B. Richter

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

Report at HEAC 1971

The discovery in the early '60's at the Princeton-Stanford ring of what was thought to be the resistive wall instability brought the realization that circular accelerators are fundamentally unstable devices because of the interaction of the beam with its environment. Stability is achieved only through Landau damping and/or some external damping system.

CBX layout (1962)



- 1965, Priceton-Stanford CBX: First mention of an 8-pole magnet
  - Observed vertical resistive wall instability
  - With octupoles, increased beam current from ~5 to 500 mA
- CERN PS: In 1959 had 10 octupoles; not used until 1968
  - At 10^{12} protons/pulse observed (1^{st} time) head-tail instability. Octupoles helped.
  - Once understood, chromaticity jump at transition was developed using sextupoles.
  - More instabilities were discovered; helped by octupoles and by feedback.

How to make a high-intensity machine?
(OR, how to make a high-intensity beam stable?)

1. Landau damping – the beam's "immune system". It is related to the spread of betatron oscillation frequencies. The larger the spread, the more stable the beam is against collective instabilities.

2. External damping (feed-back) system – presently the most commonly used mechanism to keep the beam stable.
   - Can not be used for some instabilities (head-tail)
   - Noise
   - Difficult in linacs

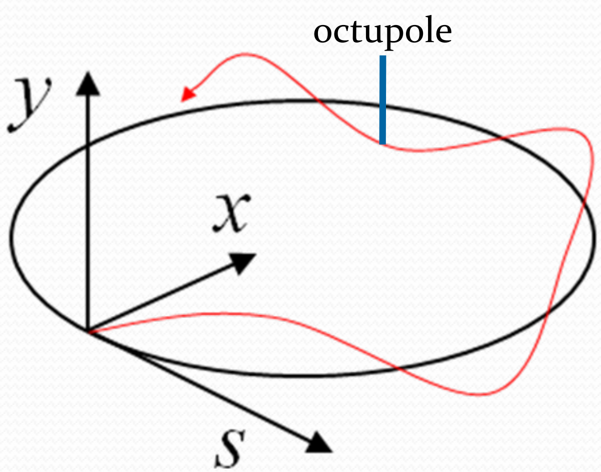
Most accelerators rely on both

- LHC
  - Has a transverse feedback system
  - Has 336 Landau Damping Octupoles
  - Provide tune spread of 0.001 at 1-sigma at injection
- In all machines there is a trade-off between Landau damping and dynamic aperture.
  - ...But it does not have to be.

Today's talk will be about…

- … How to improve beam's immune system (Landau damping through betatron frequency spread)
  - Tune spread not ~0.001 but 10-50%
- **What can be wrong with the immune system?**
  - The main feature of all present accelerators – particles have nearly identical betatron frequencies (tunes) by design. This results in two problems:
    I. Single particle motion can be unstable due to resonant perturbations (magnet imperfections and non-linear elements);
    II. Landau damping of instabilities is suppressed because the frequency spread is small.

To create the tune spread, we add non-linear elements (octupoles) as best we can.



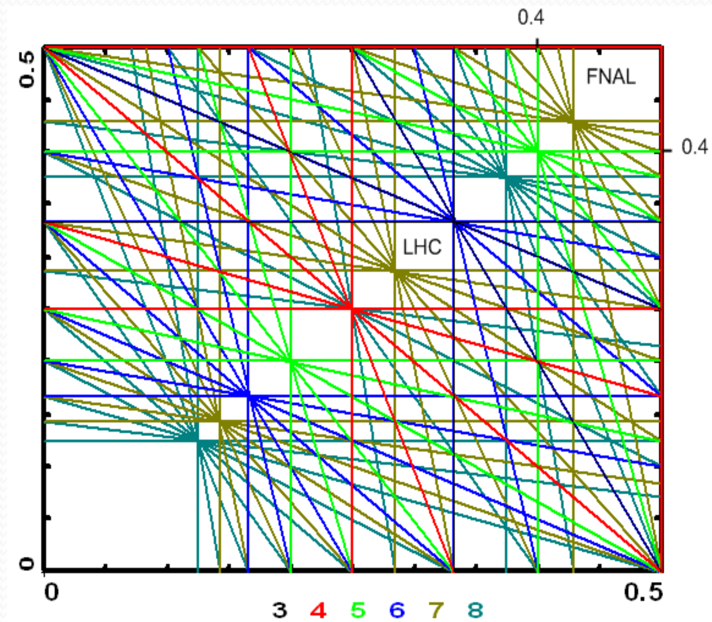
Tune spread depends on a linear tune location

1-D system:

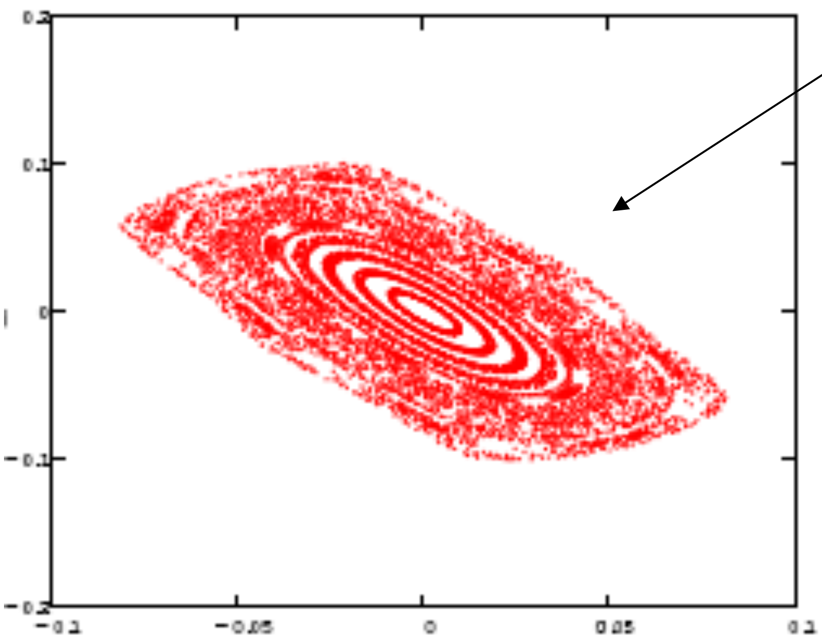
Theoretical max. spread is 0.125

2-D system:

Max. spread < 0.05



1 octupole in a linear 2-D lattice



Typical phase space portrait:

1. Regular orbits at small amplitudes
2. Resonant islands + chaos at larger amplitudes;

Are there “magic” nonlinearities that create large spread and zero resonance strength?

The answer is – yes
(we call them “integrable”)

$$H = F(J_x, J_y)$$

Courant-Snyder Invariant

Equation of motion for betatron oscillations

$$z'' + K(s)z = 0,$$

$$z = x \text{ or } y$$

- Courant and Snyder found a conserved quantity:

$$J = \frac{1}{2\beta(s)}\left(z^2 + \left(\frac{\beta'(s)}{2}z - \beta(s)z'\right)^2\right)$$

where $\left(\sqrt{\beta}\right)'' + K(s)\sqrt{\beta} = \frac{1}{\sqrt{\beta^3}}$ -- auxiliary equation

$$H(J_x, J_y) = \omega_x J_x + \omega_y J_y$$

J_x, J_y -- are Courant-Snyder integrals of motion

$\omega_x = \frac{\partial H}{\partial J_x}$ $\omega_y = \frac{\partial H}{\partial J_y}$ -- betatron frequencies

Linear function of actions: good or bad?

$$H(J_x, J_y) = \omega_x J_x + \omega_y J_y$$

- It is convenient (to have linear optics), easy to model, ...but it is NOT good for stability.
- We did not know (until now) how to make it any other way!

First non-linear accelerator proposals (before KAM theory)

- In a series of reports 1962-65 Yuri Orlov has proposed to use non-linear focusing as an alternative to strong (linear) focusing.
  - Final report (1965):

FUNDAMENTAL PROPERTIES OF NON-LINEAR FOCUSING*

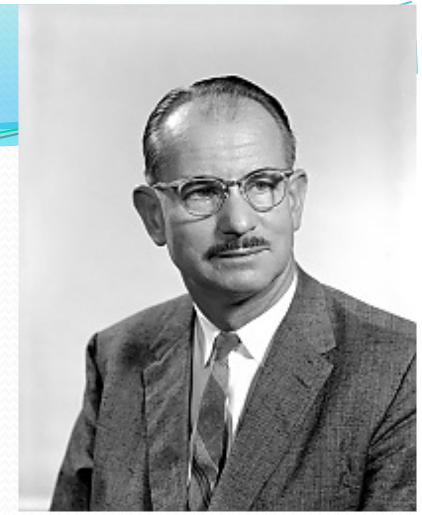
V. V. Vecheslavov and Yu. F. Orlov

(*Received* 23 *July* 1965)

Abstract—An analysis has been made of the fundamental properties of non-linear focusing taking the simple example of non-linear focusing in a symmetric magnetic field of the fifth degree. The dimensions of the first stability region with regard to small non-linear z-oscillations are determined. The influence of r–z-resonances was studied and also the maintenance of stability when allowing for adiabatic damping with the help of external or mutual r- and z-phase stabilization. It was found that mutual phase stabilization arises in the region of a r–z-resonance.

A numerical and partly analytical study of these effects has been made.

McMillan nonlinear optics



- In 1967 E. McMillan published a paper

SOME THOUGHTS ON STABILITY
IN NONLINEAR PERIODIC FOCUSING SYSTEMS

Edwin M. McMillan

September 5, 1967

- Final report in 1971. This is what later became known as the "McMillan mapping":

$$x_i = p_{i-1}$$
$$p_i = -x_{i-1} + f(x_i)$$

$$f(x) = -\frac{Bx^2 + Dx}{Ax^2 + Bx + C}$$

$$Ax^2 p^2 + B\left(x^2 p + xp^2\right) + C\left(x^2 + p^2\right) + Dxp = \text{const}$$

If $A = B = 0$ one obtains the Courant-Snyder invariant

McMillan 1D mapping

$$f(x) = -\frac{Bx^2 + Dx}{Ax^2 + Bx + C}$$

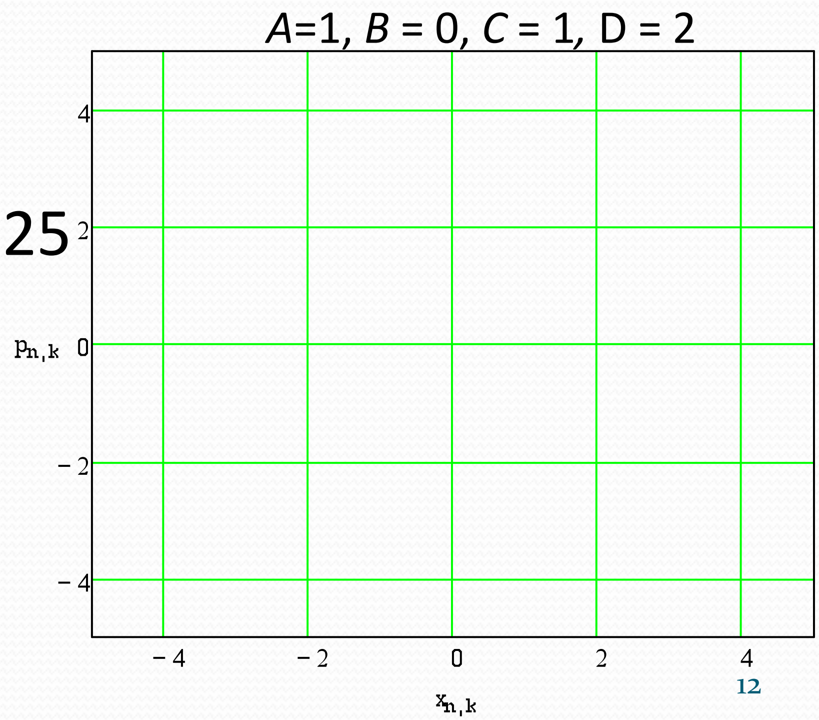
- At small *x*: $f(x) \to -\dfrac{D}{C}x$

  Linear matrix: $\begin{pmatrix} 0 & 1 \\ -1 & -\dfrac{D}{C} \end{pmatrix}$ Bare tune: $\dfrac{1}{2\pi}\operatorname{acos}\left(-\dfrac{D}{2C}\right)$

- At large *x*: $f(x) \to 0$

  Linear matrix: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Tune: 0.25

- Thus, a tune spread of 50-100% is possible!



What about 2D optics?

- How to extend McMillan mapping into 2-D?
- Danilov, Perevedentsev found two 2-D examples:
  - Round beam: $xp_y - yp_x = const$
  1. Radial McMillan kick: $r/(1 + r^2)$ -- Can be realized with an "Electron lens" or in beam-beam head-on collisions
  2. Radial McMillan kick: $r/(1 - r^2)$ -- Can be realized with solenoids (may be useful for linacs)
- In general, the problem is that the Laplace equation couples x and y fields of the non-linear thin lens

Are all integrable systems stable?

- No, some are better than others

Long-term stability

- The first paper on the subject was written by Nikolay Nekhoroshev in 1971

Russian Math. Surveys 32:6 (1977), 1–65
From Uspekhi Mat. Nauk 32:6 (1977), 5–66

AN EXPONENTIAL ESTIMATE OF THE TIME OF STABILITY OF NEARLY-INTEGRABLE HAMILTONIAN SYSTEMS

N. N. Nekhoroshev

1.1 Nearly-integrable Hamiltonian systems. Perpetual stability and stability during finite intervals of time. In this article we investigate the behaviour of the variables I in the Hamiltonian system of canonical equations

$$\dot{I} = -\frac{\partial H}{\partial \varphi}, \quad \dot{\varphi} = \frac{\partial H}{\partial I}$$

with the Hamiltonian

(1.1) $$H = H_0(I) + \varepsilon H_1(I, \varphi),$$

where $\varepsilon \ll 1$ is a small parameter, the perturbation $\varepsilon H_1(I, \varphi)$ is 2π-periodic in $\varphi = \varphi_1, \ldots, \varphi_s$, and I is an s-dimensional vector, $I = I_1, \ldots, I_s$.

- He proved that for sufficiently small ε $\quad \|I(t) - I(0)\| \leq R_* \epsilon^b \quad \text{for} \quad |t| \leq T_* \exp\left(\epsilon^{-a}\right)$
  provided that $H_o(I)$ meets certain conditions know as steepness
  - Convex and quasi-convex functions $H_o(I)$ are the steepest
- An example of a NON-STEEP function is a linear function

$$H_0(I_1, I_2) = \nu_1 I_1 + \nu_2 I_2$$

- Another example of a NON-STEEP function is

$$H_0(I_1, I_2) = I_1^2 - I_2^2$$

Non-linear Hamiltonians

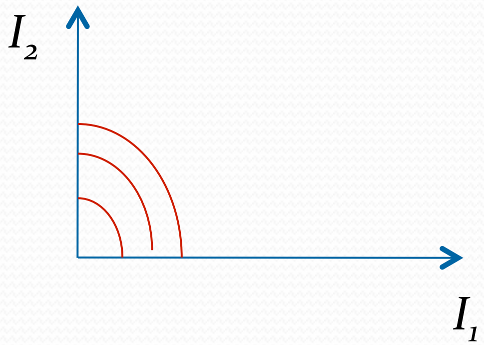
- We were looking for (and found) non-linear 2-D steep Hamiltonians that can be implemented in an accelerator
- Other authors worked on this subject recently: J. Cary, W. Wan et al., S. Danilov, E. Perevedentsev
  - The problem in 2-D is that the fields of non-linear elements are coupled by the Laplace equation.
  - An example of a steep (convex) Hamiltonian is

$$H_0(I_1, I_2) = \alpha_1 I_1^2 + \alpha_2 I_2^2,\ \alpha > 0$$

but we DO NOT know how to implement it with magnetic fields…

What are we looking for?

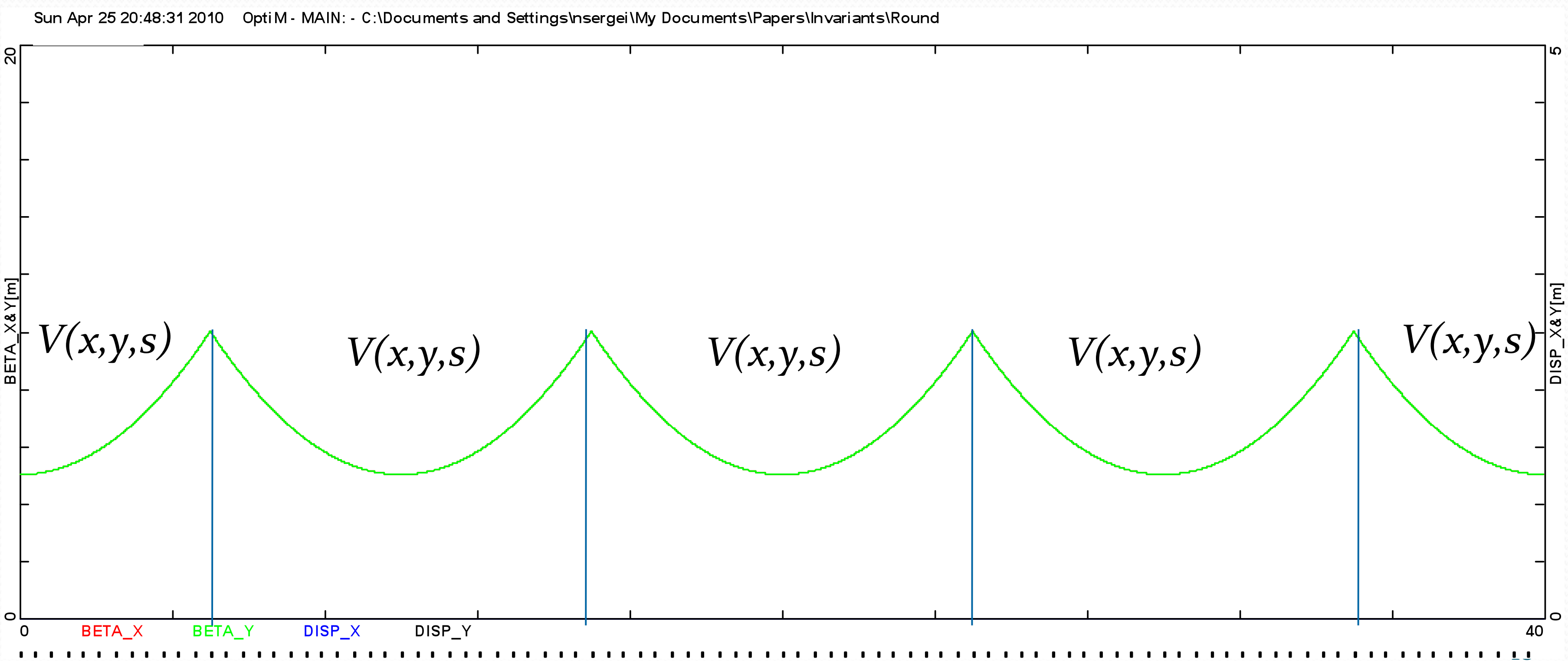
- We are looking for a 2-D integrable convex non-linear Hamiltonian, $H_0(I_1, I_2) = h(I_1, I_2)$
  - $h(I_1, I_2) = const$ -- convex curves



Our approach

- See: Phys. Rev. ST Accel. Beams 13, 084002
- Start with a round axially-symmetric LINEAR focusing lattice (FOFO)
  - Add special non-linear potential *V(x,y,s)* such that

$$\Delta V(x,y,s) \approx \Delta V(x,y) = 0$$



Special time-dependent potential

- Let's consider a Hamiltonian

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + K(s)\left(\frac{x^2}{2} + \frac{y^2}{2}\right) + V(x, y, s)$$

where *V(x,y,s)* satisfies the Laplace equation in 2d:

$$\Delta V(x, y, s) \approx \Delta V(x, y) = 0$$

- In normalized variables we will have:

$$z_N = \frac{z}{\sqrt{\beta(s)}},$$

$$p_N = p\sqrt{\beta(s)} - \frac{\beta'(s)z}{2\sqrt{\beta(s)}},$$

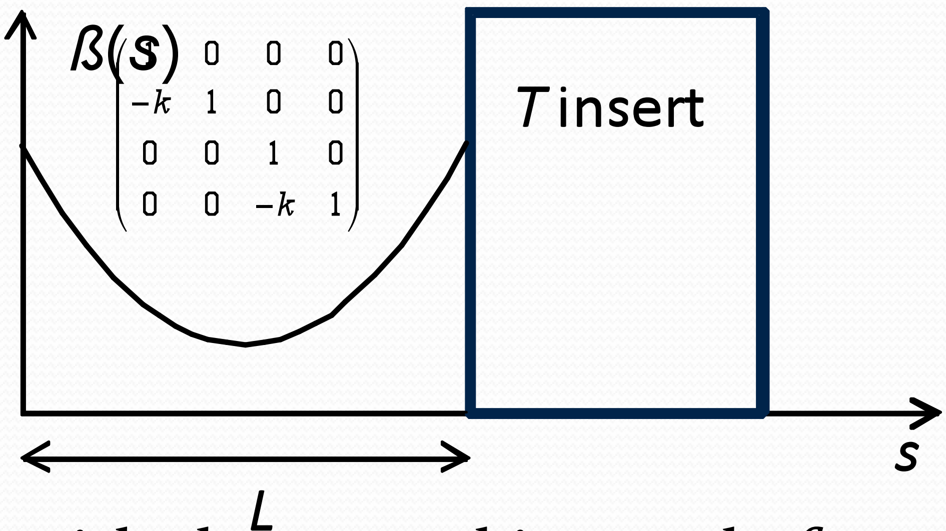
$$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + \beta(\psi)V\left(x_N\sqrt{\beta(\psi)}, y_N\sqrt{\beta(\psi)}, s(\psi)\right)$$

Where new "time" variable is $\psi(s) = \int_0^s \frac{ds'}{\beta(s')}$

Four main ideas

1. Chose the potential to be time-independent in new variables $$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + U(x_N, y_N)$$
2. Element of periodicity

$$\beta(s) = \frac{L - sk(L-s)}{\sqrt{1 - \left(1 - \frac{Lk}{2}\right)^2}}$$



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -k & 1 \end{pmatrix}$$

3. Find potentials *U(x, y)* with the second integral of motion
4. Convert Hamiltonian to action variables $H_0(I_1, I_2) = h(I_1, I_2)$ and check it for steepness

How to make the Hamiltonian time-independent?

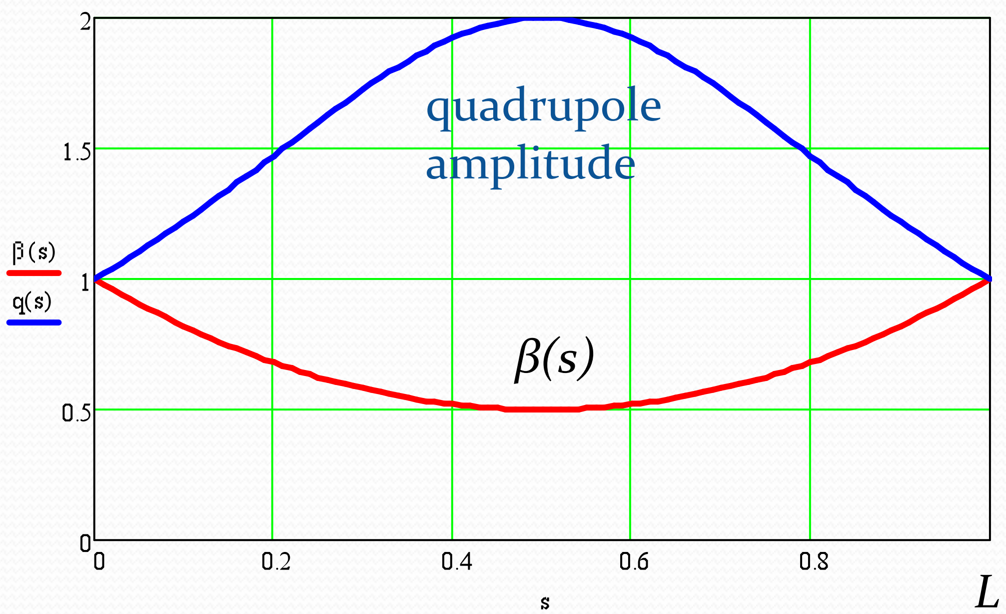
$$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + \beta(\psi) V\left(x_N \sqrt{\beta(\psi)}, y_N \sqrt{\beta(\psi)}, s(\psi)\right)$$

- Example: quadrupoles

$$V(x, y, s) = \frac{q}{\beta(s)^2}\left(x^2 - y^2\right)$$

$$U(x_N, y_N) = q\left(x_N^2 - y_N^2\right)$$

$$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + q\left(x_N^2 - y_N^2\right)$$



Integrable but still linear…

Tunes: $\nu_x^2 = \nu_0^2(1 + 2q)$

$\nu_y^2 = \nu_0^2(1 - 2q)$

Tune spread: zero

Integrable 2-D Hamiltonians

- Look for second integrals quadratic in momentum
  - All such potentials are separable in some variables (cartesian, polar, elliptic, parabolic)
  - First comprehensive study by Gaston Darboux (1901)
- So, we are looking for integrable potentials such that

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2}{2} + U(x, y)$$

Second integral: $$I = Ap_x^2 + Bp_x p_y + Cp_y^2 + D(x, y)$$

$$A = ay^2 + c^2,$$

$$B = -2axy,$$

$$C = ax^2,$$

Darboux equation (1901)

- Let $a \neq 0$ and $c \neq 0$, then we will take $a = 1$

$$xy\left(U_{xx} - U_{yy}\right) + \left(y^2 - x^2 + c^2\right)U_{xy} + 3yU_x - 3xU_y = 0$$

- General solution

$$U(x,y) = \frac{f(\xi) + g(\eta)}{\xi^2 - \eta^2}$$

$$\xi = \frac{\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2}}{2c}$$

$$\eta = \frac{\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}}{2c}$$

ξ : [1, ∞], η : [-1, 1], f and g arbitrary functions

The second integral

- The 2nd integral

$$I\left(x, y, p_x, p_y\right) = \left(xp_y - yp_x\right)^2 + c^2 p_x^2 + 2c^2 \frac{f(\xi)\eta^2 + g(\eta)\xi^2}{\xi^2 - \eta^2}$$

- Example: $$U(x, y) = \frac{1}{2}\left(x^2 + y^2\right)$$

$$f_1(\xi) = \frac{c^2}{2}\xi^2\left(\xi^2 - 1\right) \qquad g_1(\eta) = \frac{c^2}{2}\eta^2\left(1 - \eta^2\right)$$

$$I\left(x, y, p_x, p_y\right) = \left(xp_y - yp_x\right)^2 + c^2 p_x^2 + c^2 x^2$$

Laplace equation

- Now we look for potentials that also satisfy the Laplace equation (in addition to the Darboux equation):

$$U_{xx} + U_{yy} = 0$$

- We found a family with 4 free parameters (b, c, d, t):

$$f_2(\xi) = \xi\sqrt{\xi^2 - 1}\left(d + t\,\mathrm{acosh}(\xi)\right)$$

$$g_2(\eta) = \eta\sqrt{1-\eta^2}\left(b + t\,\mathrm{acos}(\eta)\right)$$

$$U(x,y) = \frac{f(\xi) + g(\eta)}{\xi^2 - \eta^2}$$

The most interesting: d=0 and $b = -\frac{\pi}{2}t$

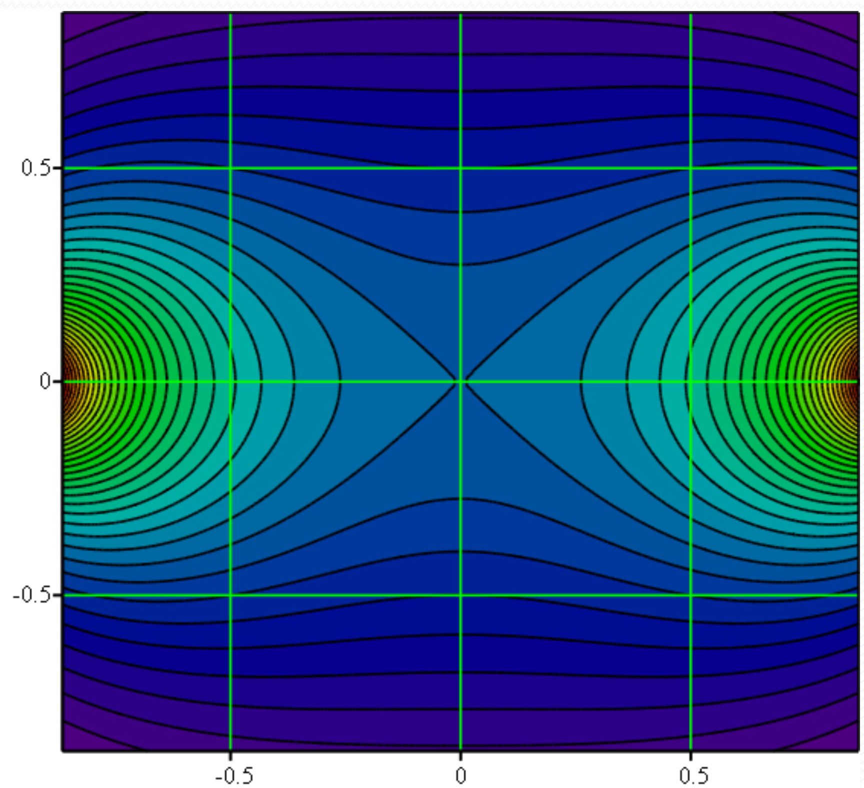
The integrable Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2}{2} + tU(x, y)$$

This potential has two adjustable parameters:
t – strength and c – location of singularities

Multipole expansion (electrostatic case):

For |z| < c $U(x,y) \approx \frac{t}{c^2}\mathrm{Re}\left((x+iy)^2 + \frac{2}{3c^2}(x+iy)^4 + \frac{8}{15c^4}(x+iy)^6 + \frac{16}{35c^6}(x+iy)^8 + \ldots \right)$



For c = 1

$|t| < 0.5$ to provide linear stability for small amplitudes

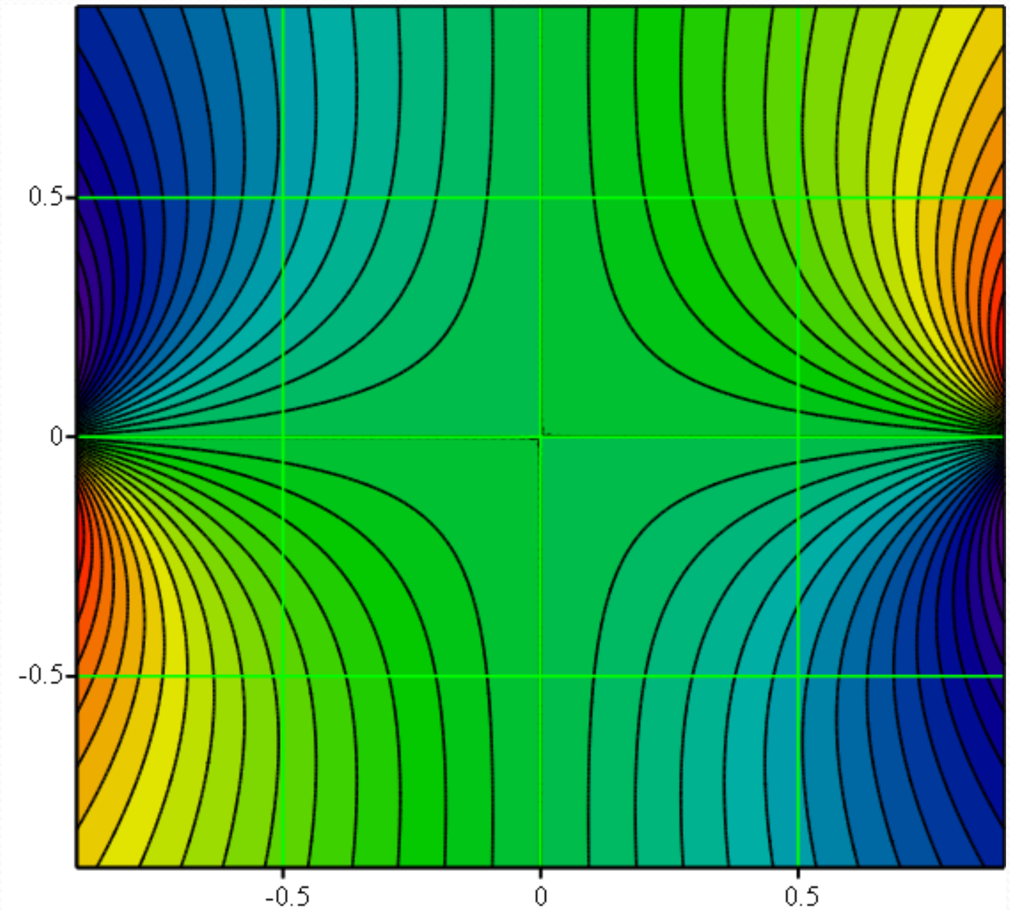
For $t > 0$ adds focusing in x

Small-amplitude tune s:

$$\nu_1 = \sqrt{1+2t}$$

$$\nu_2 = \sqrt{1-2t}$$

- Magnetostatic case



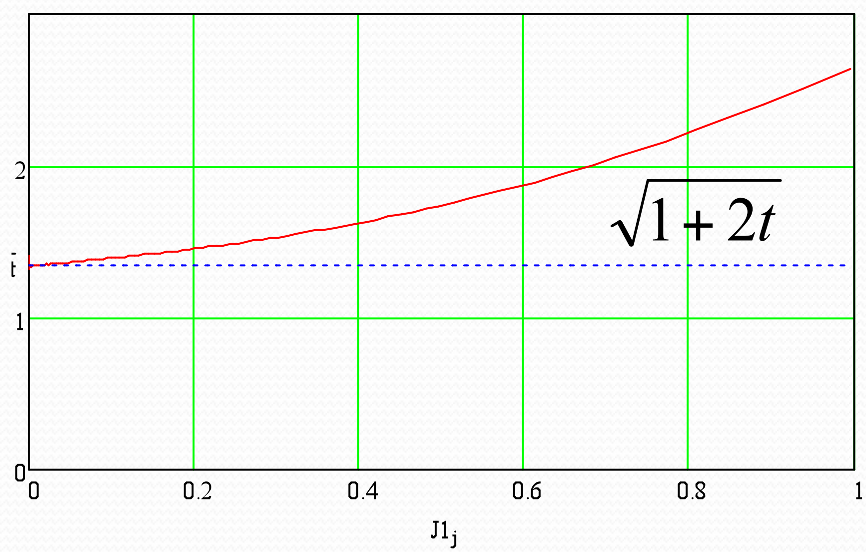
B

For |z| < c $$U(x,y) \approx \frac{t}{c^2}\,\mathrm{Im}\left((x+iy)^2 + \frac{2}{3c^2}(x+iy)^4 + \frac{8}{15c^4}(x+iy)^6 + \frac{16}{35c^6}(x+iy)^8 + \ldots\right)$$

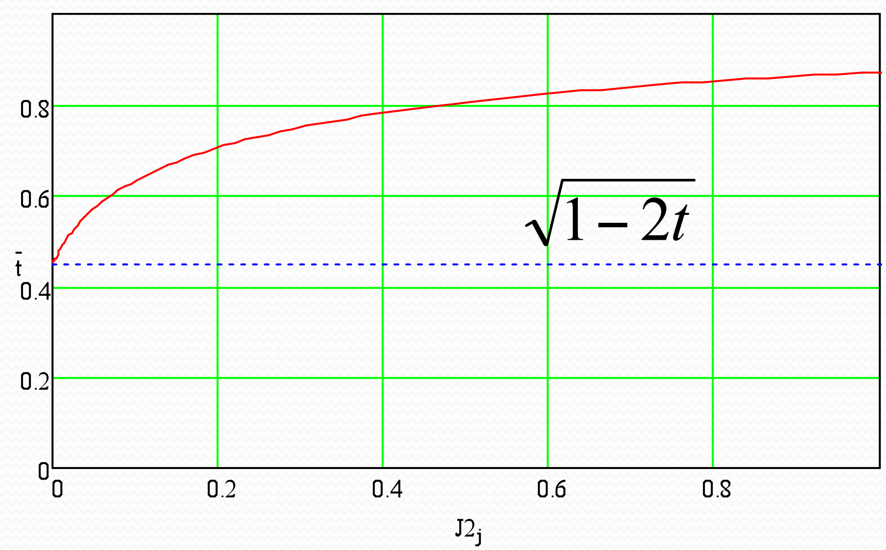
Convex Hamiltonian

- This Hamiltonian is convex (steep)
- Example of tunes for t = 0.4

$\nu_1(J_1, 0)$



$\nu_2(0, J_2)$

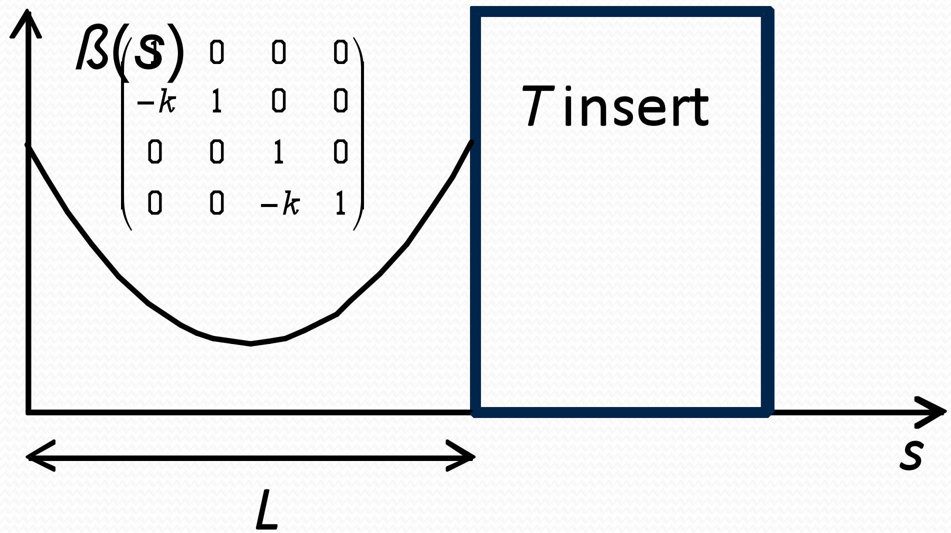


For t -> 0.5 tune spreads of ~ 100% is possible

How to realize it?

- Need to create an element of periodicity.
  - The T-insert can also be

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ k & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & k & -1 \end{pmatrix}$$

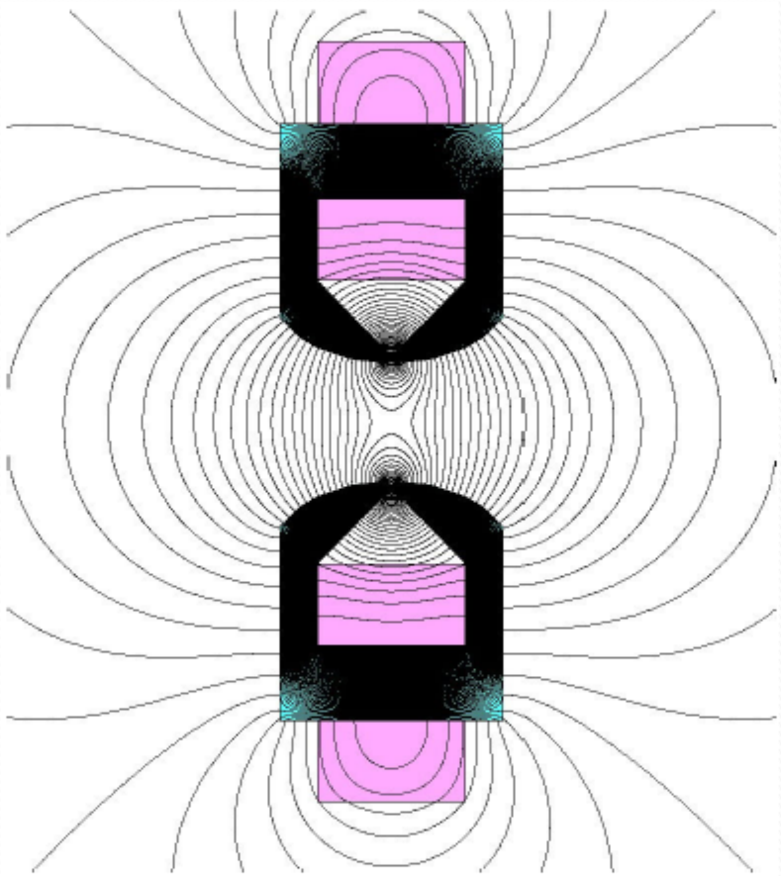


which results in a phase advance 0.5 (180 degrees) for the T-insert.

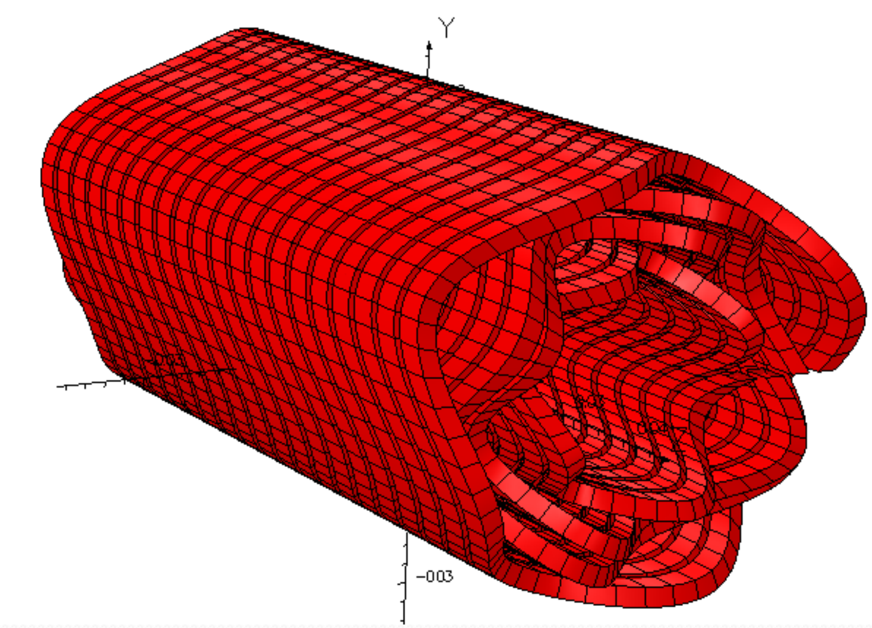
  - The drift space L can give the phase advance of at most 0.5 (180 degrees).
  - Beam size should not be small – particles need to "sample" nonlinear fields.

How to make these elements?

Proposal 1: custom-built magnet

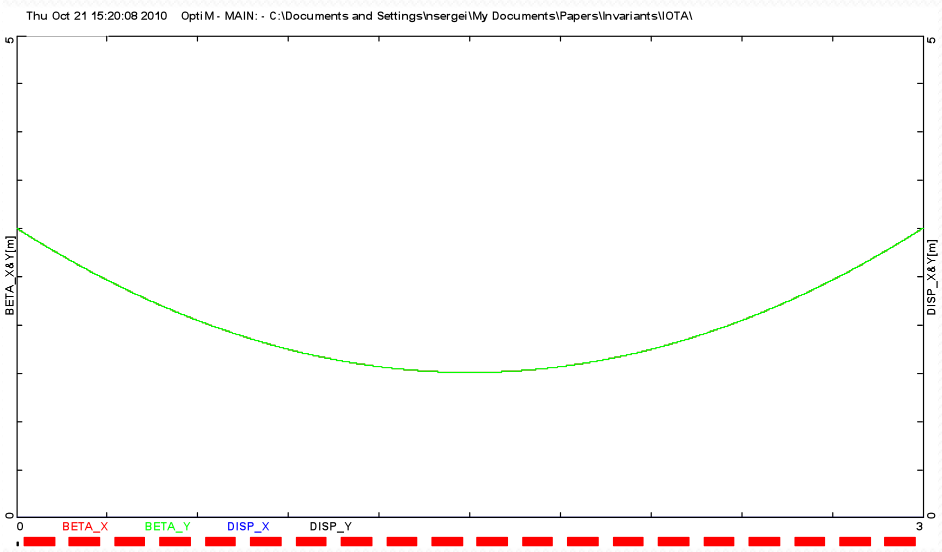


Proposal 2: multipole expansion

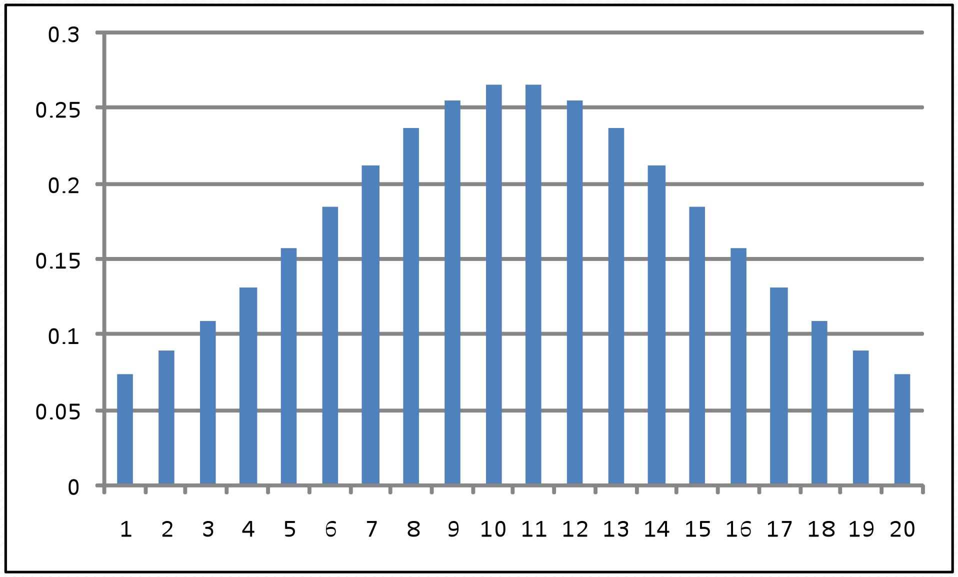


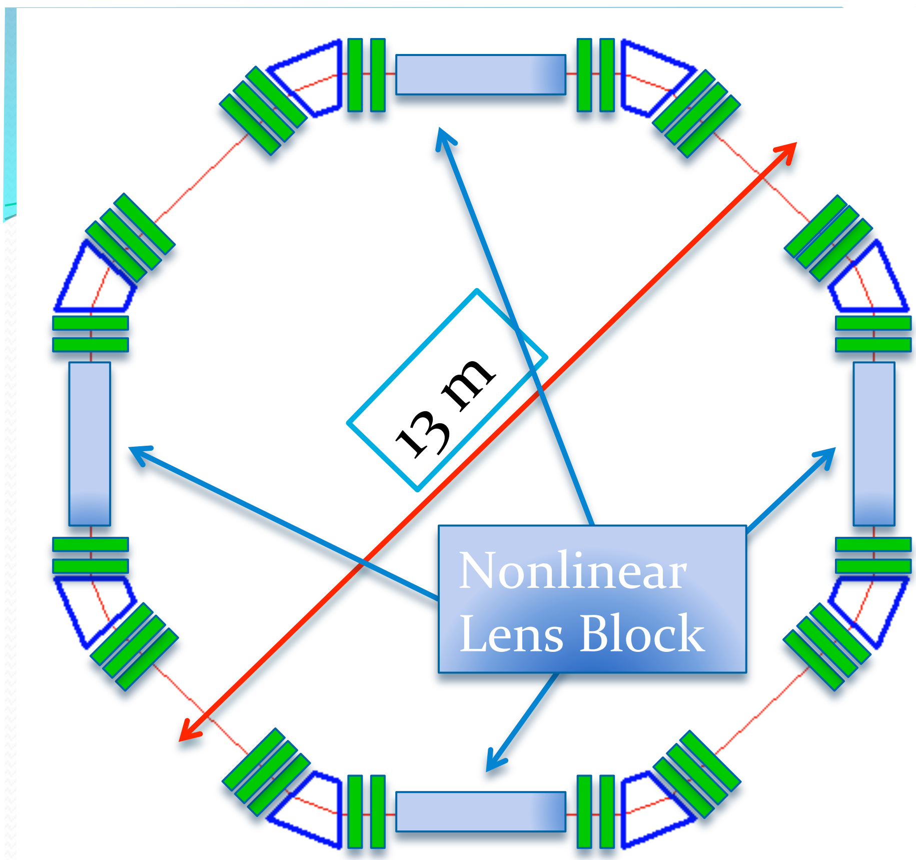
Non-linear elements section

- Number of elements per section: 20 – 30
- Section length: 2 -3 m

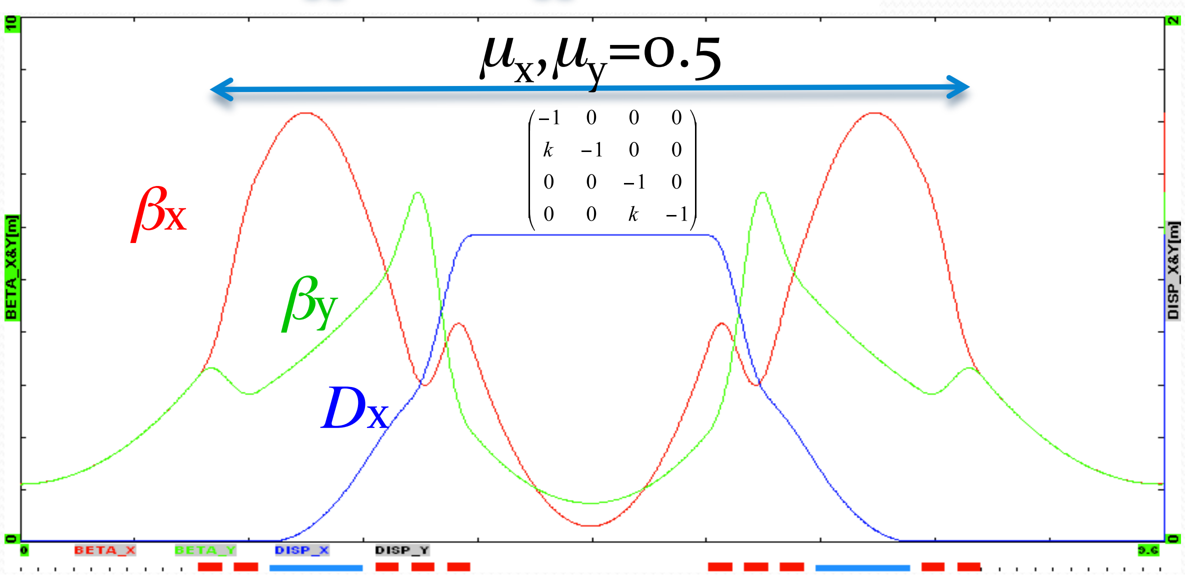


Quadrupole component strength (T/m)



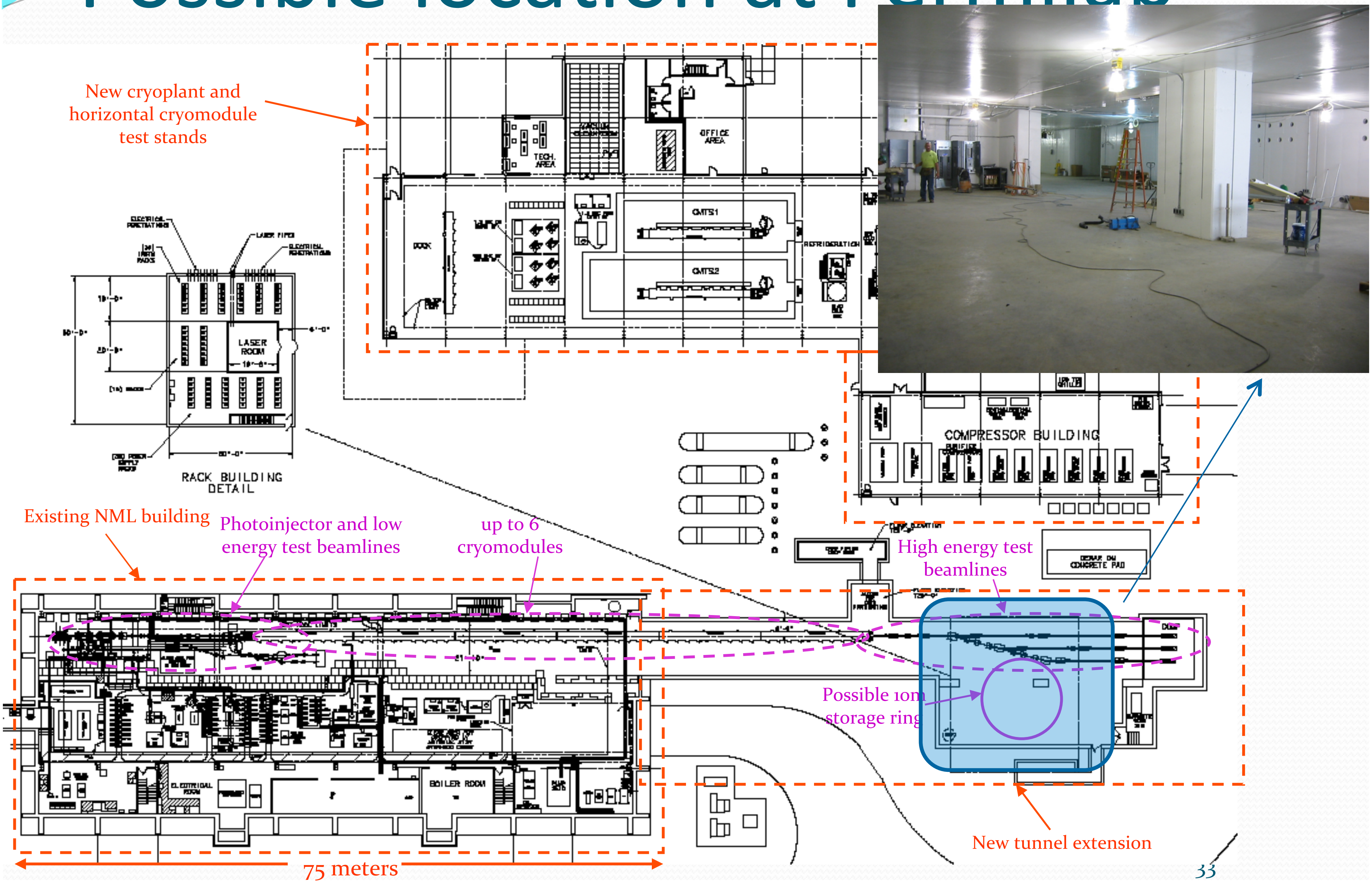


e- Energy	**150 MeV**
Circumference	38 m
Dipole field	0.5 T
Betatron tunes	Qx=Qy=3.2 (2.4 to 3.6)
Radiation damping time	1-2 s (10^7 turns)
Equilibrium emittance, rms, non-norm	0.06 μm

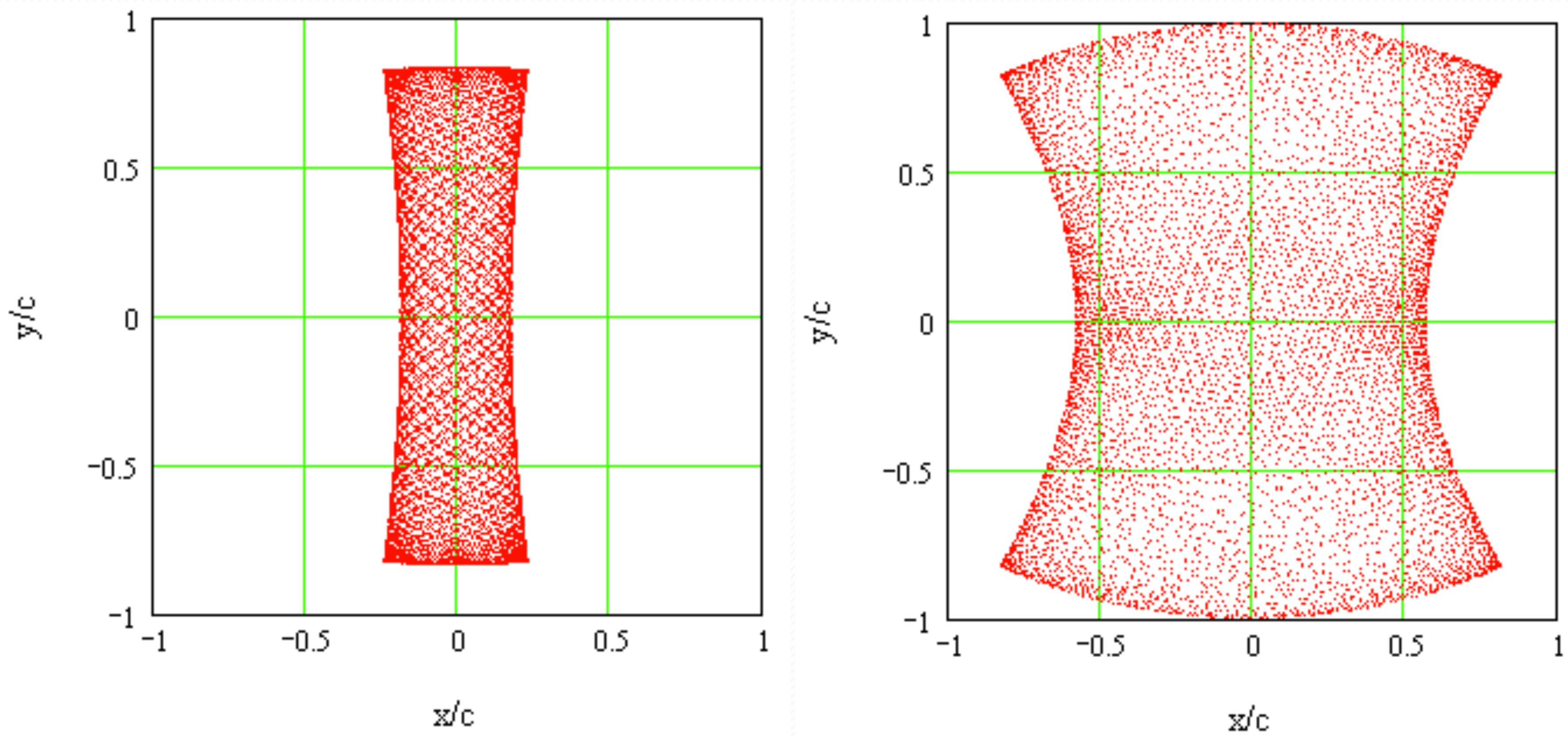


Nonlinear lens block	
Length	2.5 m
Number of elements	20
Element length	0.1 m
Max. gradient	1 T/m
Pole-to-pole distance (min)	~ 2 cm

Possible location at Fermilab

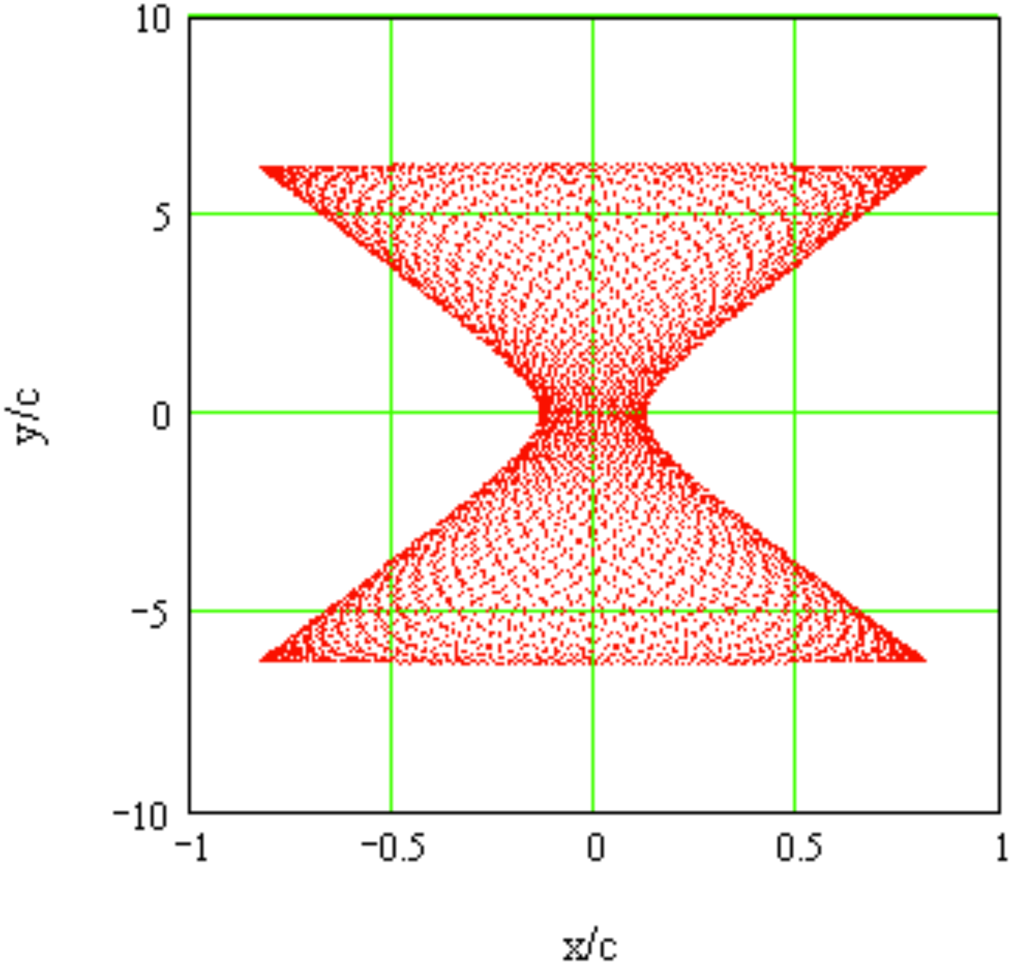


Phase space with *t*<0.5, r<c



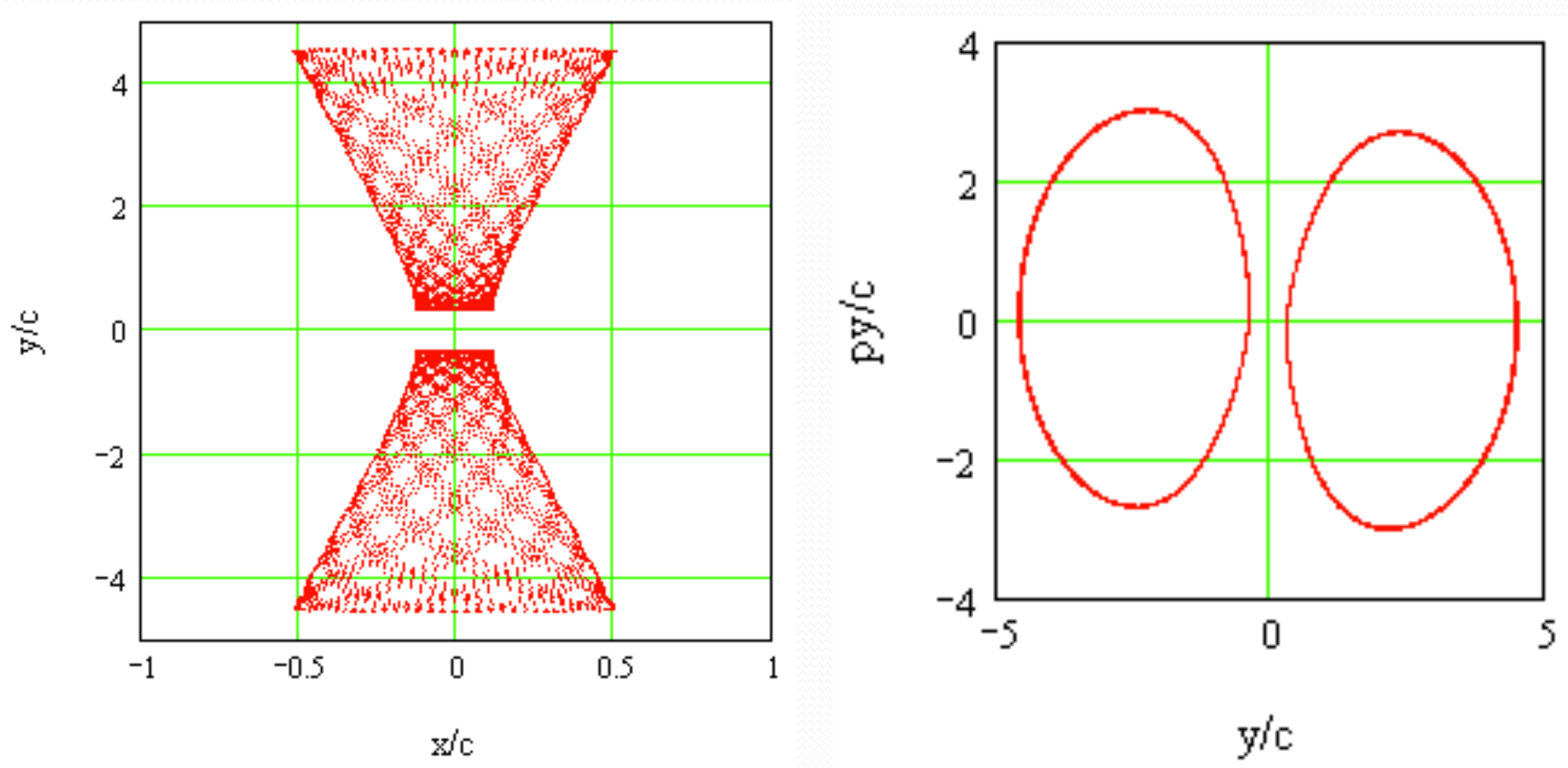
At Ax,Ay<0.5c trajectories remain inside r=c circle. Multipole expansion is valid.

Phase space with t<0.5, r>c



Motion is stable. Can not use multipole expansion of the potential.

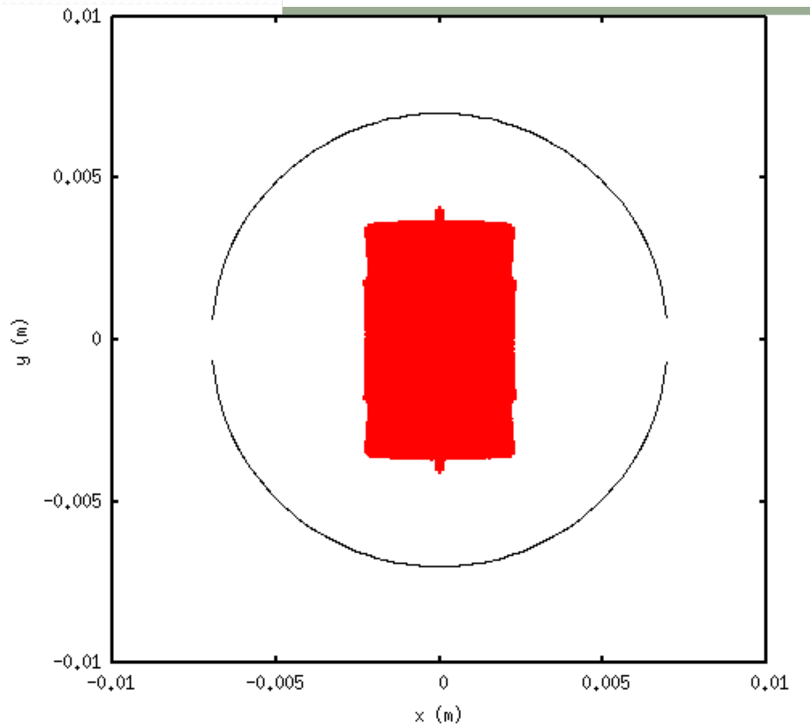
Phase space with *t*>0.5

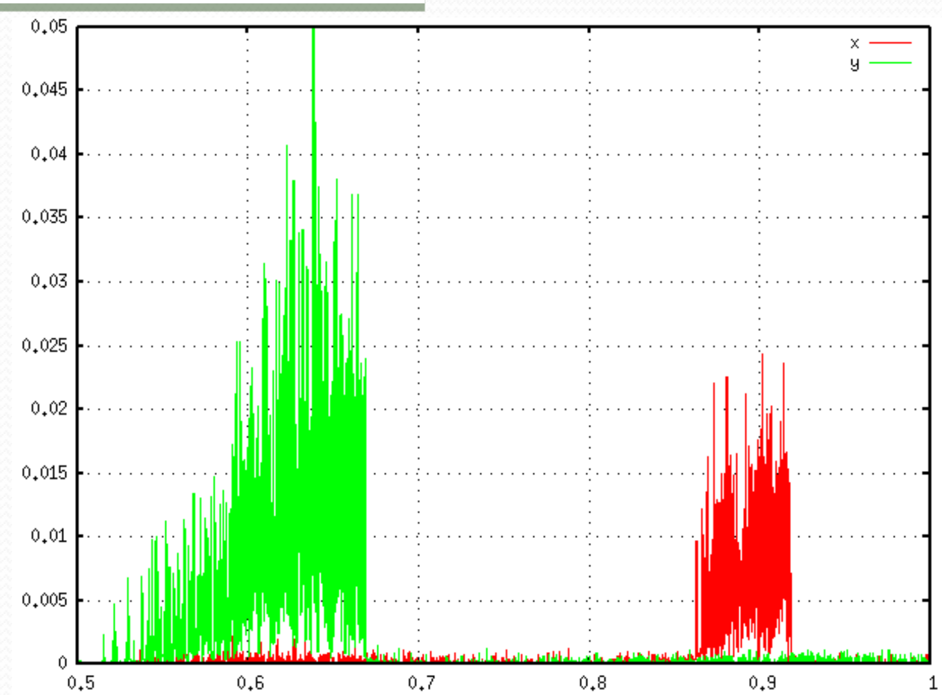


y=0 is unstable point. Still, it is possible to contain trajectories with x<c but y>c

How much tune spread?

Conservative variant: round pipe, multipole n=9





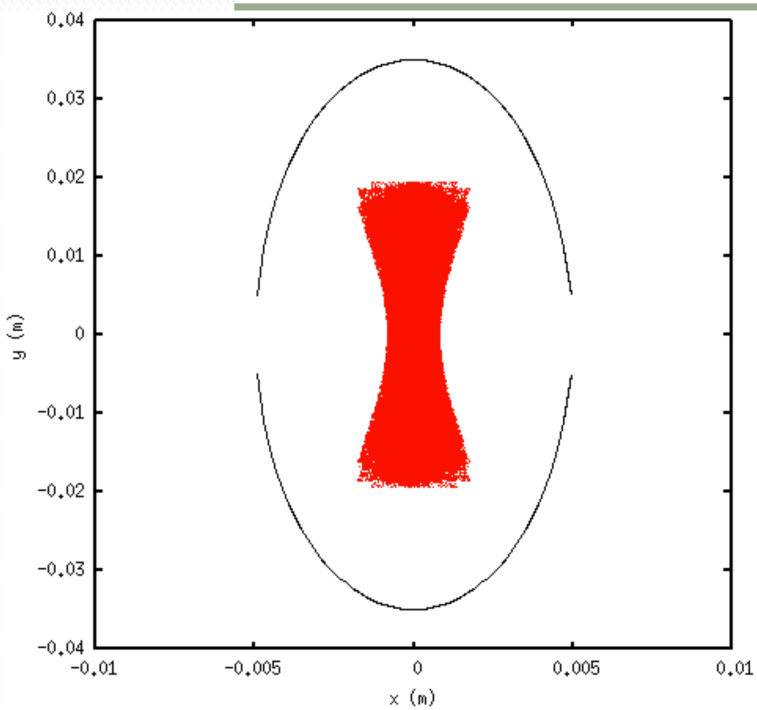
c=10 mm
pipe radius = 7 mm
beam within r=5 mm

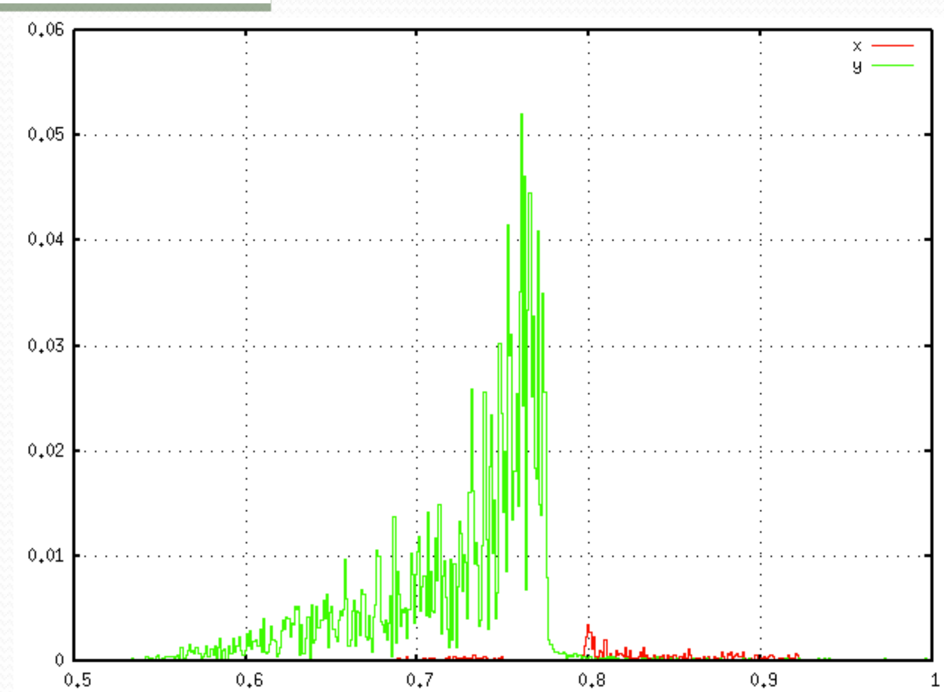
Dipole moment spectrum
tracking one cell Q_0=0.8 δQ_{max}=0.3
δQ_y=0.15, δQ_x=0.06
with 4 cells -> δQ_y=0.45, δQ_x=0.24

δQ_{max} is determined by phase advance in drift. Maximum is 0.5

How much tune spread?

Less conservative variant: ‘true’ lens, t<0.5



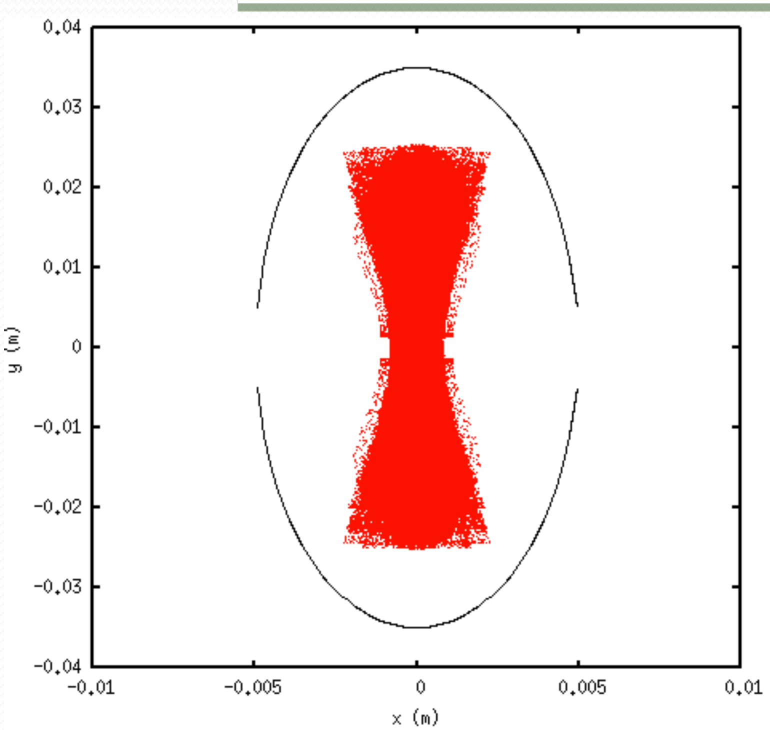


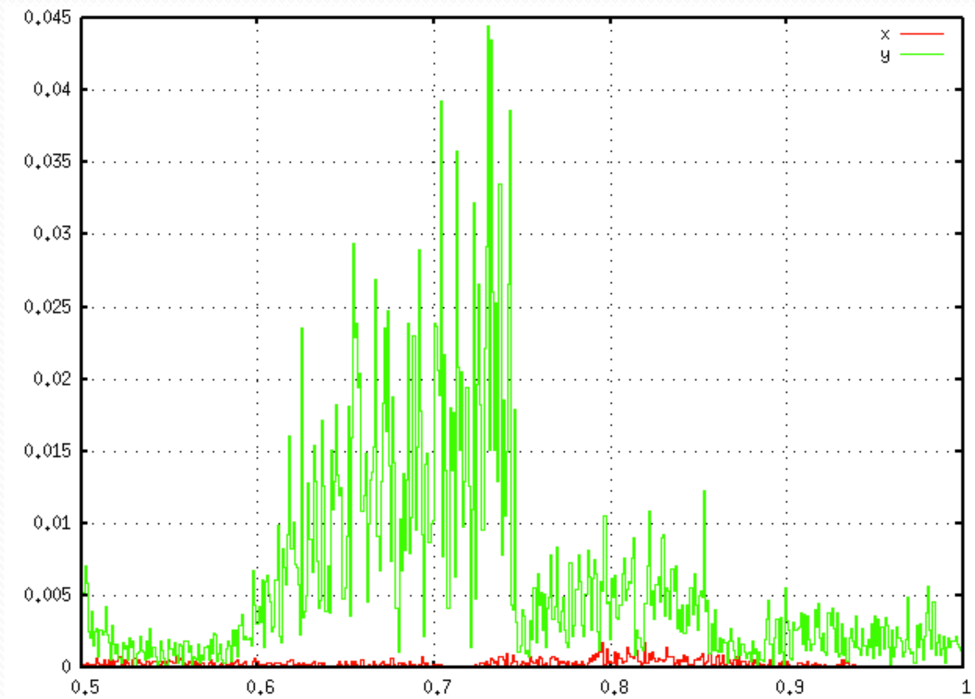
c=10 mm
pipe size x=5 y=35 mm
beam within x=2 mm

Dipole moment spectrum
tracking one cell Q_0=0.8 δQ_{max}=0.3
δQy=0.25, δQx=0.12
with 4 cells -> δQy=1, δQx=0.48

How much tune spread?

Full blast: ‘true’ lens, t=1.5



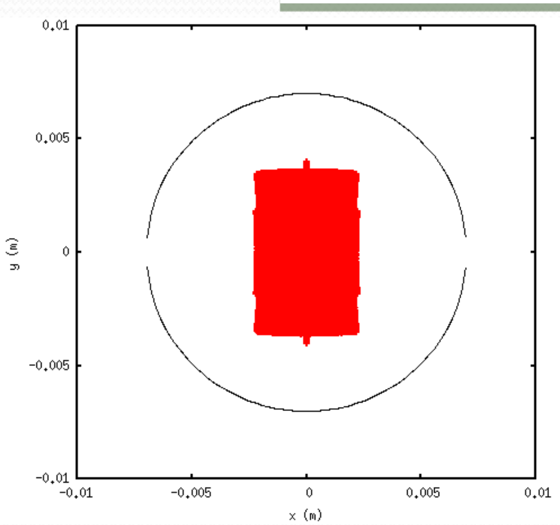


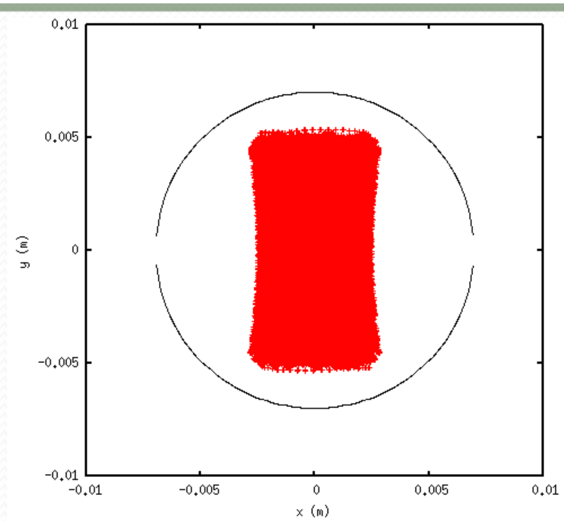
c=10 mm
pipe size x=5 y=35 mm
beam within x=2.5 mm

Dipole moment spectrum
tracking one cell Q_o=0.8 δQ_{max}=0.5
δQy=0.5, δQx=0.2
with 4 cells -> δQy=2, δQx=0.8

Effects of imperfections

Conservative variant: round pipe, multipole n=9





c=10 mm
pipe radius = 7 mm
beam within r=5 mm
$\delta Qy=0.4$, $\delta Qx=0.2$

Stability is preserved with

- Transverse misalignments r.m.s. up to 0.5 mm
- Synchrotron oscillations $\sigma_E=0.001$, C=-15
- $\beta x/\beta y$ difference up to 10%
- $\mu x \neq \mu y \neq 0.5$ up to 0.01
- Sextupoles in the arcs DAx=c

Current and Proposed Studies

Numerical Simulations

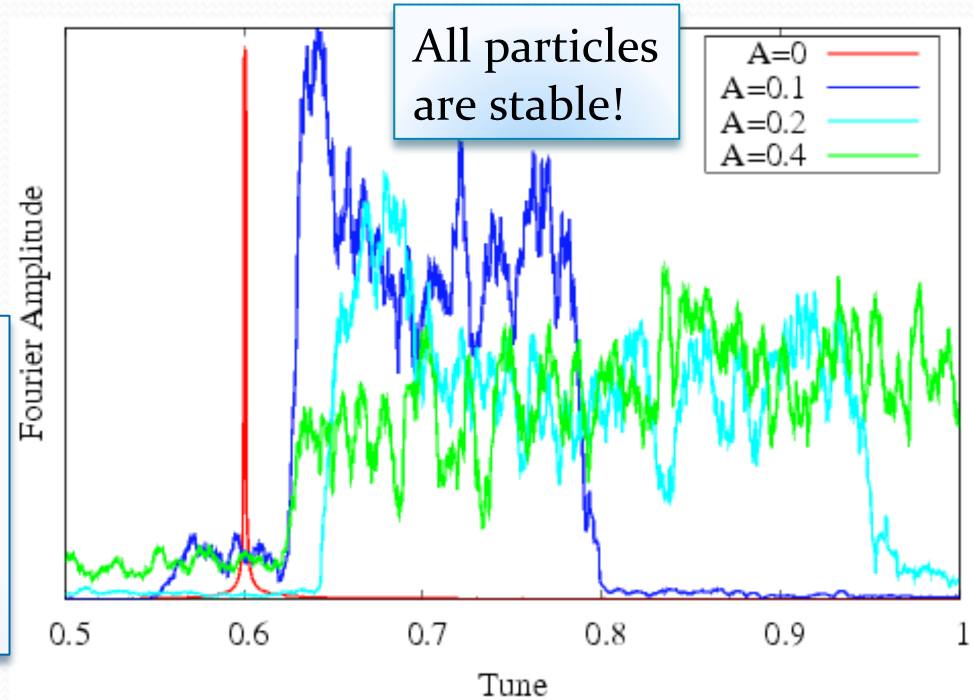
- Nonlinear lenses implemented in a multi-particle tracking code (MAD-X and PTC)
- Studied particle stability
  - Effects of imperfections (phase advance, beta-functions, etc.) - acceptable
  - Synchrotron motion - acceptable
  - Number of nonlinear lenses - 20
- Simulated observable tune spread
- To Do:
  - Ring nonlinearities
  - Chromaticity

Spectrum of horizontal dipole moment
Q_0=0.9x4=3.6
5000 particles
8000 revolutions
But up to 10^6 revolutions simulated

Possible Experiments at Test Ring

- Demonstrate large betatron tune spread
- Demonstrate part of the beam crossing integer resonance
- Map phase space with pencil beam by varying an injection error

All particles are stable!



Fermilab interests

- Academic:
  - no resonances
  - long-term stability at large amplitudes
  - large tune spreads – Landau damping
- Practical:
  - Electron machines
    - large dynamic apertures
  - Proton machines
    - super-high currents
    - instability damping

Relevant to DOE/HEP

Conclusions

- We found first examples of completely integrable non-linear optics.
  - Tune spreads of 50% are possible. In our test ring simulation we achieved tune spread of about 1.5 (out of 3.6);
- Nonlinear “integrable” accelerator optics has advanced to possible practical implementations
  - Provides “infinite” Landau damping
  - Potential to make an order of magnitude jump in beam brightness and intensity
- Fermilab is in a good position to use of all these developments for next accelerator projects
  - Rings or linacs

Acknowledgements

- Many thanks to:
  - Etienne Forest (KEK) for element implementation in PTC
  - Frank Schmidt (CERN) for optics implementation in MAD-X
  - Vladimir Kashiknin (FNAL) and Holger Witte (Oxford) for magnet proposals

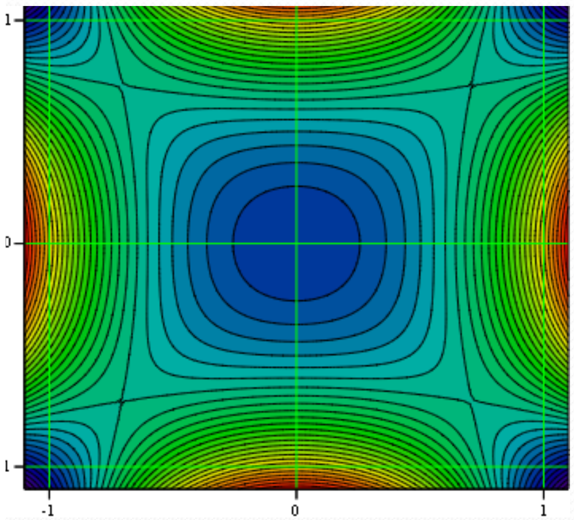
Extra slides

Example of time-independent Hamiltonians

- Octupole

$$V(x,y,s) = \frac{\kappa}{\beta(s)^3}\left(\frac{x^4}{4} + \frac{y^4}{4} - \frac{3x^2y^2}{2}\right)$$

$$U = \kappa\left(\frac{x_N^4}{4} + \frac{y_N^4}{4} - \frac{3y_N^2 x_N^2}{2}\right)$$



$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{k}{4}\left(x^4 + y^4 - 6x^2y^2\right)$$

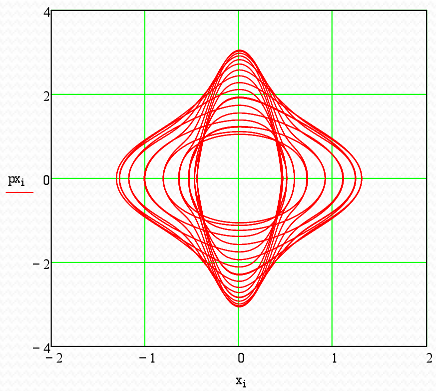
This Hamiltonian is NOT integrable

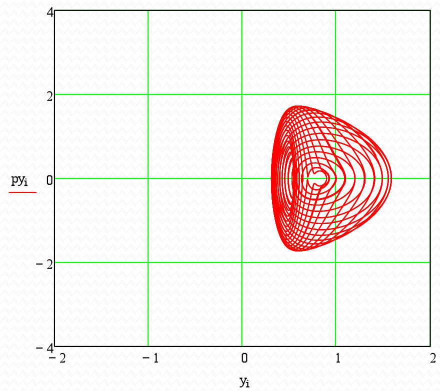
Tune spread (in both x and y) is limited to ~12%

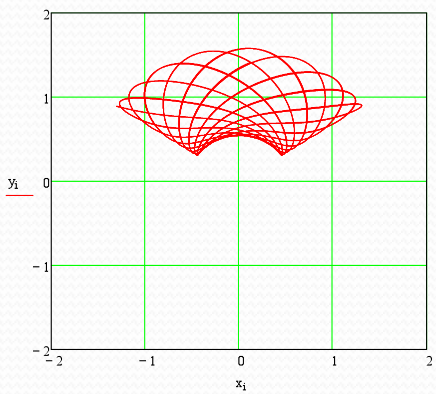
Example of exactly integrable nonlinear Hamiltonian

$$H_N = \frac{p_{xN}^2 + p_{yN}^2}{2} + \frac{x_N^2 + y_N^2}{2} + \beta(\psi) V\left(x_N \sqrt{\beta(\psi)}, y_N \sqrt{\beta(\psi)}, s(\psi)\right)$$

$$V(x,y) = U(x,y) = \frac{a\left(x^2 - y^2\right) + 2bxy}{\left(x^2 + y^2\right)^2}$$







- This gives EXACT integrability

$$I = \left(xp_y - yp_x\right)^2 + 2\frac{a\left(x^2 - y^2\right) + 2bxy}{x^2 + y^2}$$

- Not all trajectories encircle the singularity!

Spectrum of vertical dipole moment. Q_0=0.905x4=3.62